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LETTER TO THE EDITOR

Depinning transition and non-universal behaviour of defects in the two-dimensional Ising model: a unified treatment

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Abstract. We develop a unified renormalisation group (RG) framework for studying depinning and/or non-universal behaviour associated with linear defects in two-dimensional Ising models. Non-universality is characterised by a fixed line in the RG recursion relations. A generalised version of the depinning transition is proposed and a nearly exact phase diagram is calculated.

Critical behaviour near the defect planes (walls) and free surfaces in an otherwise homogeneous system has attracted considerable attention recently. In particular, in 2D Ising models, such defects have been studied in the context of surface and interface phenomena (Fisher and Ferdinand 1967, Oliveira *et al* 1978, Burkhardt and Eisenriegler 1981), the pinning–depinning transition (Abraham 1980, 1981a, b, Chalker 1981), and non-universal behaviour near the internal line of defects (McCoy and Perk 1980, Bariev 1979, Burkhardt and Eisenriegler 1981). In this note we shall consider the 2D generalised Ising model with a defect line. Our purpose is to show that the seemingly different phenomena mentioned above can all be studied quite quantitatively within the single renormalisation group (RG) framework. We will propose the most general type of depinning transition and calculate the corresponding phase diagram. (For a related study in 3D Ising models see Pandit *et al* (1982), Wortis and Švrakić (1982), Burkhardt and Eisenriegler (1981), and in particular Nakanishi and Fisher (1982).) In order to make this statement of purpose more precise it is useful first to define a model.

Consider a 2D, square, nearest-neighbour Ising model with an internal 'ladder' of defect bonds, as shown in figure 1. On one side of the defect the couplings will have

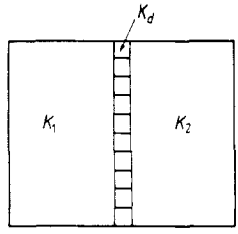

Figure 1. Schematic representation of the model studied. Defect bonds are horizontal bonds shown. All the bonds to the right of the defect have values K_2, those to the left, values K_1.

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values $K_1 = J_1/k_B T$, and on the other, values $K_2 = J_2/k_B T$. The defect bonds will have values $K_d = J_d/k_B T$. This model differs from the models usually studied by the fact that, generally, $K_1 \neq K_2$ in our case. However, if we take:

(1) periodic boundary conditions, $K_1 = K_2$, K_d arbitrary, the model is equivalent to the one studied by McCoy and Perk (1980);

(2) antiperiodic boundary conditions and, e.g., $K_1 = \infty$, $K_2 < \infty$, $K_d < K_2$, we recover the interface-pinning model of Abraham (1980);

(3) arbitrary boundary conditions with $K_d = 0$, the model reduces to two independent Ising models (K_1 and K_2) with free surfaces.

The general model considered in this work, with $K_1 \neq K_2$, is most properly thought of as the model of two thermodynamic systems interacting via K_d. Note that case (3) above is equivalent to cases (1) and/or (2) with $K_d = 0$. Before proceeding the following remarks should be relevant.

(1) For model (1) it has been exactly shown (McCoy and Perk 1980) that at bulk criticality ($K_1 = K_2 = K_c$) the exponent $\eta_\parallel$ of the spin–spin correlation function along the defect varies continuously with the defect coupling strength K_d, thus exhibiting non-universal behaviour. The RG flows should correspondingly show a fixed line (Burkhardt and Eisenriegler 1981).

(2) For model (2) it has been exactly shown (Abraham 1980) that the interface created by the boundary conditions remains pinned to the defect at low temperatures but, as the temperature is increased above a certain value $T_R(K_d)$, the interface depins. However, if the defect is internal (i.e. not located at the boundary), the interface will be pinned to it for all temperatures $T < T_c$ (this has been exactly shown by Abraham (1981b)) and the depinning transition will not occur. The physics behind the depinning transition is the following. At low temperatures, when the energy terms dominate, it is favourable for the system to have an interface pinned to the (weaker) defect couplings. At higher temperatures the entropy is gained by the depinning of the interface which is free to wander. Exactly at the depinning transition temperature $T_R(K_d)$, the defect specific heat exhibits a jump discontinuity induced by the extra degrees of freedom available from the depinned interface (Abraham 1980). Clearly, when the defect is internal (Abraham 1981b) the interface cannot 'depin' because of the symmetric situation on both sides of the defect. Our purpose in this note is to show that the depinning transition *can* take place even when the defect is internal *provided* $K_1 \neq K_2$. Physically, this means that one system (the one with the larger coupling constant) is promoting its own order inside the other system. This we consider to be the most general type of the depinning transition which, to our knowledge, has not been studied before.

The above statements can be made more precise within the RG argument, to which we turn now. In what follows we shall consider only a first-order cumulant expansion approximation (see e.g. Oliveira *et al* 1978, Burkhardt and van Leeuwen 1982). Other schemes, for reasons we do not entirely understand, fail to reproduce the physical picture described above. The recursion relations in the 3D parameter space (K_1, K_2, K_d) are

$$K_i' = 2K_i \langle S_i \rangle^2 \qquad i = 1, 2 \tag{1}$$

$$K_d' = 2K_d \langle S_1 \rangle \langle S_2 \rangle \tag{2}$$

where $\langle S_i \rangle$ is the average value of the spin in the basic 2×2 block. With the majority rule projection operator (Burkhardt and van Leeuwen 1982) we have $\langle S_i \rangle =$

$(e^{4K_i}+2)/(e^{4K_i}+6+e^{-4K_i})$, $i=1,2$, but we shall take $\langle S_i\rangle=(e^{4K_i}+p)/(e^{4K_i}+6+e^{-4K_i})$ where p is some adjustable parameter. With $p=p_e=4\sqrt{2}-3$, the exact fixed point $K=\frac{1}{2}\ln(1+\sqrt{2})$ is reproduced and we shall choose this value of parameter p in our calculation. The correlation length critical exponent ν will depend on the choice of parameter p and it has values $\nu=0.994$ (for $p=2$, i.e. with the majority rule projection) and $\nu=1.226$ (with $p=p_e=4\sqrt{2}-3$). These should be compared with the exact value $\nu=1$. Clearly, with the latter choice of parameter p the exponent value becomes worse, but for the global behaviour it is more convenient to take $p=p_e$. In particular, the phase diagram and the thermodynamic functions, calculated with $p=p_e$, differ from the known exact results by ~2% over the whole temperature range.

First, let us consider the model studied by McCoy and Perk (1980), described above. In this case, as explained above, we take $K_1=K_2$ and equations (1) reduce to a single (bulk) recursion relation which has a fixed point at

$$K^*=2K^*\langle S^*\rangle^2=\tfrac{1}{2}\ln(1+\sqrt{2}). \tag{3}$$

Note, however, that the fixed point equation (3) implies $\langle S^*\rangle=1/\sqrt{2}$, irrespectively of the choice of parameter p. When this is substituted into equation (2) we get

$$K'_d=K_d \qquad \text{(at bulk criticality)} \tag{4}$$

which is a fixed line solution, as expected (Burkhardt and Eisenriegler 1981). Of course, equation (4) contains $K_d^*=0$ and $K_d^*=K^*$ as special solutions corresponding to the free surface and the bulk problem, respectively, each having its own set of exponents. Thus, the topology of RG flows proposed by Burkhardt and Eisenriegler (1981) is recovered.

Now, consider the model (2) for the interface pinning, studied by Abraham (1980). In order to create a 'hard surface' we shall take $K_1=\infty$. Thus, we have a defect next to the fully ordered system. With this choice of K_1 we have $\langle S_1\rangle=1$, and (1) and (2) become

$$K'_2=2K_2\langle S_2\rangle^2 \tag{5}$$

$$K'_d=2K_d\langle S_2\rangle. \tag{6}$$

Equation (5) has the same fixed point solution as equation (3). Let us then consider $K_2>K^*$ (i.e. the system is in the low-temperature phase $T<T_c$). Equations (5) and (6) exhibit the following important property: since $\langle S_2\rangle\leqslant 1$, K_d will grow with iterations faster than K_2 (simply, (6) contains a first power of $\langle S_2\rangle$, while (5) contains a square of this quantity). To make contact with the work of Abraham (1980), let us take $K_d=aK_2$, where $0\leqslant a\leqslant 1$. It can be seen from (5) and (6) that, even though initially $K_d<K_2$, the flows will be such that after a certain number of iterations n, the coupling $K_d^{(n)}$ may become larger than the coupling $K_2^{(n)}$. However, this will not always happen: if the initial value of the coupling K_2 is sufficiently large (i.e. the initial temperature is sufficiently low) then $\langle S_2\rangle$ will approach unity after only a few iterations, and K_d and K_2 will subsequently grow at the same rate. Therefore, after infinitely many iterations, we can obtain two different results: (i) the coupling $K_d^{(\infty)}<K_2^{(\infty)}$, or (ii) $K_d^{(\infty)}>K_2^{(\infty)}$, corresponding to different physical situations at zero temperature. In case (i) the interface is pinned to the defect and in case (ii) the interface is depinned. The depinning transition temperature $T_R(a)$, for a given value of defect parameter a, is defined as that initial value of K_2 for which $K_d^{(\infty)}=K_2^{(\infty)}$. Numerically, the

depinning transition phase diagram is calculated simply by iterating equations (5) and (6):

$$K_2^{(N)} = 2^N K_2 \prod_{n=0}^{N-1} \langle S_2^{(n)} \rangle^2 \tag{7}$$

$$K_d^{(N)} = 2^N a K_2 \prod_{n=0}^{N-1} \langle S_2^{(n)} \rangle \tag{8}$$

thus giving (by equating equations (7) and (8))

$$a = \lim_{N \to \infty} \prod_{n=0}^{N} \langle S_2^{(n)} \rangle \tag{9}$$

which is convergent since $\langle S_2^{(n)} \rangle \to 1$ as $n \to \infty$, where we have taken $\langle S_2^{(n)} \rangle = (\exp 4K_2^{(n)} + p_e)/(\exp 4K_2^{(n)} + 6 + \exp -4K_2^{(n)})$, the nth iterate of $\langle S_2 \rangle$. Figure 2 shows the depinning transition phase diagram obtained from (9). For comparison we also show the exact phase diagram (Abraham 1980), which is given by

$$a = (2K)^{-1} \ln\{(\cosh 2K - e^{-2K} \sinh 2K) \pm [(\cosh 2K - e^{-2K} \sinh 2K)^2 - 1]^{1/2}\}. \tag{10}$$

The agreement is quite satisfactory since the result (9) and the exact result (10) differ by not more than 2% over the whole temperature range.

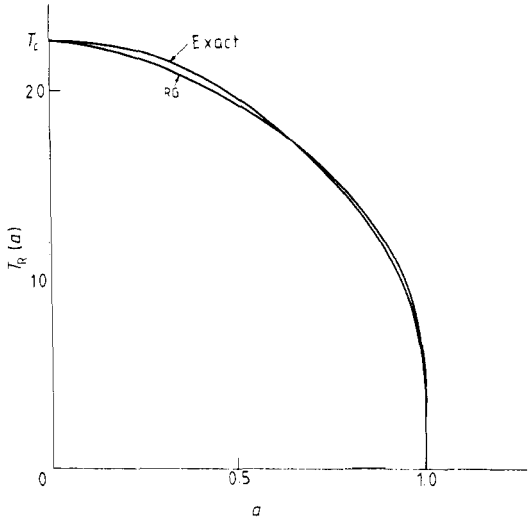

Figure 2. Depinning transition phase diagram obtained from RG calculation, equation (9), labelled by 'RG' in the figure. For comparison, the exact phase diagram, equation (10), is also shown.

Two remarks are in order. First, equation (10) has two solutions symmetric about $a = 0$, corresponding to wetting and drying transitions (Pandit *et al* 1982). The same is true of equation (9). This is, of course, the consequence of the fact that one can use Abraham's model with *periodic* boundary conditions, but with the inverted sign of parameter a, and obtain the same results. Second, note that the interface tension

σ is given (Oliveira *et al* 1978) by

$$\sigma = 2K_2 \lim_{N\to\infty} \prod_{n=0}^{N} \langle S_2^{(n)} \rangle^2 \tag{11}$$

implying, quite generally, within the RG scheme, that

$$a = (\sigma/2K_2)^{1/2} \tag{12}$$

which is an intriguing result. The relation (12) is almost exactly satisfied by the exact solutions. That is, if we plot the exact equation (10) and the expression (12) substituting the exact answer (Fisher and Ferdinand 1967) for the interface tension $\sigma = 2K_2 + \ln \tanh K_2$, then the two curves are so close to each other that they are graphically indistinguishable. But, they differ numerically on the fourth decimal place at intermediate temperatures. The implication of (12) is that, as $a \to 0$, $T_R(a) \to T_c$ with the power $\mu/2$ ($= \frac{1}{2}$ for the $d = 2$ Ising model, which can also be derived from (10)) since $\sigma \propto (T_c - T)^{\mu}$ (Widom 1972).

It should be noted that the depinning transition temperature $T_R(a)$ is not obtained from the fixed point in recursion relations but from the global character of the RG flows. The jump in the defect specific heat (Abraham 1980) is induced by the change from $K_d^{(n)} < K_2^{(n)}$ behaviour to $K_d^{(n)} > K_2^{(n)}$ behaviour in the course of RG iterations.

Now, the general case of the depinning transition can be analysed by considering the complete set of equations (1) and (2). Let us take $K_d < K_2 < K_1$ as the initial values and iterate equations (1) and (2). Furthermore, suppose that $K_1, K_2 > K$ (i.e. both systems, K_1 and K_2, are in the ordered phase). Because of the inequality

$$\langle S_1 \rangle^2 \geqslant \langle S_1 \rangle \langle S_2 \rangle \geqslant \langle S_2 \rangle^2, \tag{13}$$

which holds for the above initial choice of couplings, the defect coupling K_d will grow faster than the (smaller) coupling K_2. The mechanism is much like the one described in discusssing equations (5) and (6). After a sufficient number N of RG iterations one either obtains $K_d^{(N)} < K_2^{(N)}$ or $K_d^{(N)} > K_2^{(N)}$, indicating that the interface is pinned to the defect (in the first case) or depinned from the defect (in the second case). The depinning transition takes place when $K_d^{(\infty)} = K_2^{(\infty)}$, i.e. by using expressions analogous to (7) and (8), we obtain

$$a = \lim_{N\to\infty} \prod_{n=0}^{N} (\langle S_2^{(n)} \rangle / \langle S_1^{(n)} \rangle) \tag{14}$$

where we have used $K_d - aK_2$, as before. Note that (14) has acceptable behaviour (i.e. $a \leqslant 1$) only if $\langle S_2^{(n)} \rangle \leqslant \langle S_1^{(n)} \rangle$ which always holds for the initial choice $K_1 > K_2$. Physically, this means that the interface depins into the phase with the smaller coupling constant (of course, the symmetric situation is obtained when the indices '1' and '2' are exchanged). Clearly, when $K_1 = \infty$, (14) reduces to (9) since $\langle S_1^{(n)} \rangle = 1$. The phase diagram resulting from (14) is shown in figure 3. In the same way as (12) was derived from (9) we can now obtain

$$a = [(\sigma_2/2K_2)(2K_1/\sigma_1)]^{1/2} \tag{15}$$

where σ_1 and σ_2 are the interface tensions of systems K_1 and K_2, respectively. We believe that (15) is a nearly exact result for the generalised depinning transition phase diagram proposed in this work (see the discussion surrounding (12)). In figure 4 we

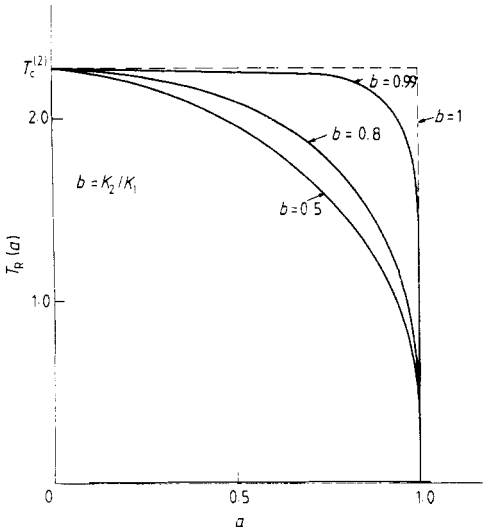

Figure 3. Depinning transition phase diagram for the general model shown in figure 1. The ratio of the two couplings K_2/K_1 (with $K_2 < K_1$) is given by a parameter b. When $b = 0$ ($K_2 = \infty$) the diagram from figure 2 is recovered. The interface is pinned in the regions under the curves. Note that when $b = 1$ the interface is always pinned.

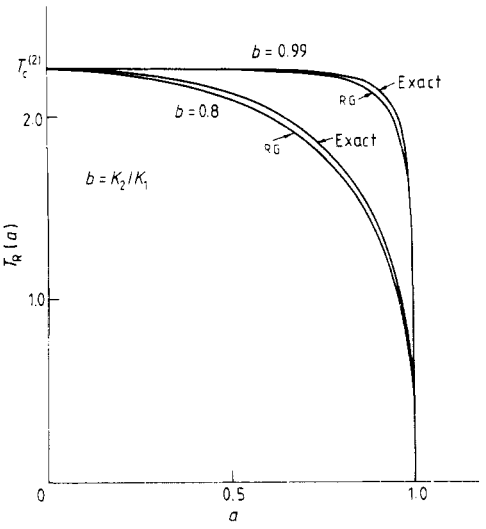

Figure 4. Same as figure 3. For comparison, the nearly exact result, equation (15), is also shown, labelled by 'Exact' (see discussion following (15)).

plot the phase diagram obtained from (15) using the exact expression (Fisher and Ferdinand 1967) $\sigma_i = 2K_i + \ln \tanh K_i$ $(i = 1, 2)$ for the interface tensions. This we label by 'exact' in figure 4 and compare it with the result (14) shown in figure 3. The agreement is again quite good (~2%) over the whole temperature range. Note that the phase diagram exhibits quite reasonable physical behaviour: as the difference between couplings K_1 and K_2 becomes smaller, the region where the interface is depinned (region above the curves) reduces, and vanishes as $K_1 \to K_2$. Numerically, when $K_1 = K_2 = K$, the expression (13) becomes an equality, implying that $K_d^{(\infty)} < K^{(\infty)}$, whenever initially $K_d < K$. This means that in the symmetric situation ($K_1 = K_2$) the (weakened) defect bonds will pin an interface for all temperatures and the depinning will not occur, in agreement with the exact result of Abraham (1981b).

We wish to emphasise that boundary conditions were never explicitly utilised in our calculation. Indeed, the systems K_1 and K_2 can both be in the same phase (ferromagnetic phase 'up', for example), and the only difference between them is that one system is more ordered than the other. At the depinning transition the more ordered system promotes its own order within the less ordered system. Of course, one can consider a situation in which one of the systems is disordered (see e.g. Lipowsky 1982), but our recursion relations cannot be applied to this problem. The reason is that the RG flows depend crucially on the value of parameter p when either of the two systems is in the disordered phase (in the low-temperature, $T < T_c$, region, considered in this work, *qualitative* behaviour is independent of parameter p, for all reasonable values of this quantity). In particular, with different choices of parameter p one obtains *qualitatively* different physical behaviour in the disordered phase and, at present, we have no criterion to decide which value is to be chosen.

The special case $K_d = 0$ has been studied via the RG method elsewhere in the context of surface behaviour (see e.g. Švrakić *et al* 1980) and we shall not consider it here. One should note, however, that the singularity in the surface specific heat at $T = T_c$ can be identified with the depinning transition at $a = 0$ (see figure 2). Similar behaviour can also be seen in $d = 3$ Ising models (Nakanishi and Fisher 1982, Švrakić *et al* 1980, Švrakić 1979). The method presented in this work can also be used for calculation of defect thermodynamic functions, and can be extended to calculations in 3D systems. This will be reported elsewhere (Švrakić 1982).

In summary, we have developed a unique RG framework for analysing depinning and/or non-universal behaviour associated with defects in Ising models. We have reproduced the expected RG flow topology for the non-universal behaviour (Burkhardt and Eisenriegler 1981). The depinning transition phase diagram has been obtained with an accuracy of ~2%. In addition, we have proposed a more general type of the depinning transition and the corresponding phase diagram is obtained with satisfactory precision. A nearly exact phase diagram is obtained when the relation (15) is used. In all cases, the depinning transition temperature is obtained from the global character of the RG flows, rather than from the special fixed point. This, we believe, is a noteworthy novel feature.

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